Volumes With Known Cross Sections

Materials Needed: modeling clay or dough, engineering graph paper, plastic knives
One problem often found on the AP Calculus free response section of the exam is finding the volume of a region with a known cross section. If you can visualize the shape, the volume is easy.

Volume of a solid over a given region with cross sections taken perpendicular to the *x*-axis

- **Semicircle cross sections –**
  1. Sketch the graphs of the given functions and determine the desired region.
  2. Mold a piece of dough or clay into a semicircular cross section on top of the region.
  3. Slice the dough or clay into slices. Each of these slices will be in the shape of a semicircle. Estimate the area of one side of each slice $\left(A = \frac{\pi r^2}{2} \right)$.
  4. Multiply each area by the approximate thickness of each slice to estimate the volume.
  5. Generalize to an integral with thickness d*x* using *x*-values for limits of integration.

Volume of a solid over a given region with cross sections taken perpendicular to the *y*-axis

- **Rectangular cross sections with a fixed height –**
  1. Sketch the graphs of the given functions and determine the desired region.
  2. Mold a piece of dough or clay into a rectangular cross section on top of the region.
  3. Slice the dough or clay into slices. Each of these slices will be in the shape of a rectangle. Estimate the area of one side of each slice (multiply the distance between the curves by the height of the rectangle to find the area) .
  4. Multiply each area by the approximate thickness of each slice to estimate the volume.
  5. Generalize to an integral with thickness d*y* using *y*-values for limits of integration.

Example:
Find the volume of the solid whose base is bounded by the graphs of $y = x + 2$ and $y = x^2$ with the indicated cross sections taken perpendicular to the *x*-axis.

semicircles –

Radius of each semicircle = $\frac{x+2-x^2}{2}$.

Area of one side of each semicircle = $\frac{\pi}{2}\left(\frac{x+2-x^2}{2}\right)^2$.

Total volume = $\frac{\pi}{2}\int_{-1}^{2}\left(\frac{x+2-x^2}{2}\right)^2 dx = \frac{81\pi}{80}$.

rectangles of height 2 –

Area of one side of each rectangle = $2\left(x+2-x^2\right)$.

Total volume = $\int_{-1}^{2} 2\left(x+2-x^2\right)dx = 9$.